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STEADY HEAT CONDUCTION FROM AN INFINITE ROW OF HOLES IN A HALF-SPACE OR A UNIFORM SLAB

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Abstract—A special boundary integral equation is used to analyze steady heat conduction from an infinite row of circular holes located below the surface of a half-space or along the midplane of a uniform slab. Numerical results show that standard handbook approximations for the total heat flux from any one hole can give sizable errors in both problems, even for modest values of the problem parameters. An alternative approximation proposed recently for the half-space problem is shown to offer a substantial improvement in accuracy.

NOMENCLATURE

a radius of the hole
d depth of holes in half-space or slab thickness
g_0, g_{1m}, g_{2m} special functions defined by equation (8) ($m = 1, 2, \ldots$)
g' special function defined by equation (5)
g'_0 special function defined by equation (10)
G defined by equation (4)
l spacing between holes
q_0, q_{1m}, q_{2m} coefficients in the harmonic expansion of the boundary flux on the hole ($m = 1, 2, \ldots$)
$\mathscr{R}, \partial\mathscr{R}$ region and its outer boundary, respectively
$\mathbf{x}, \mathbf{y}$ points in the region

Greek symbols
θ polar angle centered at ξ
ξ center of the hole
ϕ temperature
$\phi_0, \phi_{1m}, \phi_{2m}$ coefficients in the harmonic expansion of the boundary temperature on the hole ($m = 1, 2, \ldots$)

1. INTRODUCTION

IN A RECENT paper, Barone and Caulk [1] proposed a system of special boundary integral equations for solving Laplace's equation in 2-dimensional regions with circular holes. In this approach, boundary quantities are expanded in circular harmonics on the holes, and each unknown coefficient is determined by a special boundary integral equation. As more harmonics are retained in the solution representation on the hole boundary, a natural sequence of approximate equations emerges from the general system. These are called the mth-order equations, where m refers to the number of non-trivial harmonics in the solution representation on the hole. When the region has only one hole, it is possible to construct a single exact equation which retains all the harmonics on the hole but involves only the solution on the outer boundary [2, 3]. This is called the single-hole equation. The effect of the hole in this equation is carried through a special kernel function that depends on both the location and radius of the hole. The results on the hole can be recovered from the solution to this equation by a simple quadrature over the outer boundary.

Approximate expressions are available in standard handbooks [4] for the total heat flux from any one of an infinite number of identical holes which are equally spaced a fixed distance below the surface of a half-space or along the midplane of a uniform slab. All the boundaries are isothermal, with the same temperature on each hole and equal temperatures on both surfaces of the slab. For the half-space, Barone and Caulk [1] showed that the handbook approximation is identical to the solution of the zeroth-order integral equations discussed above. Based on this result, they offered the analytical solution of the first-order equations as an improvement over the handbook approximation. Although these two expressions can give significantly different results over a moderate range of the problem parameters, no estimate was given in ref. [1] for the relative error associated with either approximation.

In the present paper, the periodicity of the solution in both the half-space and the slab is exploited to solve both problems numerically using the exact single-hole equation. These results are then used to evaluate the accuracy of the approximate expressions discussed above. It turns out that the handbook expressions for both problems can produce errors greater than 20%, even for modest values of the problem parameters. On the other hand, the error associated with the new approximation for the half-space [1] is always less than 1% as long as the centers of the holes are more than one diameter from the surface and two diameters apart.

Numerical results are also given for the case when one surface of the slab is completely insulated. No simple approximate expression is available for this case.

2. BASIC INTEGRAL EQUATIONS

Consider a two-dimensional region $\mathscr{R}$ containing a circular hole of radius a, centered at $\mathbf{x} = \xi$. Let ϕ be the temperature in $\mathscr{R}$ and let ϕ and its outward normal

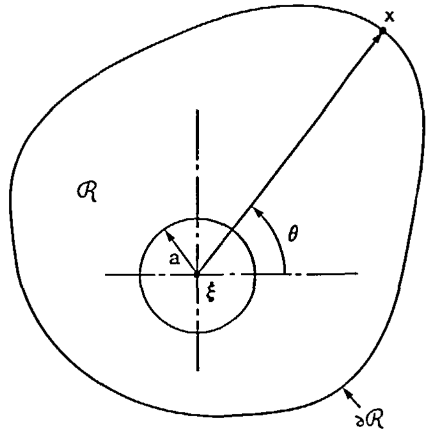

FIG. 1. Geometry and notation.

derivative on the hole be represented by

$$\phi = \phi_0 + \sum_{m=1}^{\infty} (\phi_{1m} \sin m\theta + \phi_{2m} \cos m\theta), \tag{1}$$

$$\frac{\partial \phi}{\partial n} = q_0 + \sum_{m=1}^{\infty} (q_{1m} \sin m\theta + q_{2m} \cos m\theta) \tag{2}$$

where ϕ_0, $\phi_{\lambda m}$, q_0, $q_{\lambda m}$ ($\lambda = 1, 2$; $m = 1,2,\ldots$) are constants and θ is the polar angle centered at ξ, measured relative to the x_1-axis (Fig. 1).

When temperature is specified on the boundary of the hole, one can show [3] that the entire solution inside $\mathscr{R}$ can be determined by solving the integral equation

$$G(\mathrm{y}) + \int_{\partial\mathscr{R}} \left(\phi \frac{\partial g'}{\partial n} - g' \frac{\partial \phi}{\partial n} \right) \mathrm{d}s - \phi_0 \log|\mathrm{y}-\xi|/\log a - \sum_{m=1}^{\infty} \frac{a^m}{|\mathrm{y}-\xi|^m} [\phi_{1m} \sin m\theta(\mathrm{y}) + \phi_{2m} \cos m\theta(\mathrm{y})] = 0 \tag{3}$$

where $\partial\mathscr{R}$ is the outer boundary of $\mathscr{R}$ and

$$G(\mathrm{y}) = \begin{cases} \phi(\mathrm{y}) & \text{when } \mathrm{y} \in \mathscr{R}, \\ \tfrac{1}{2}\phi(\mathrm{y}) & \text{when } \mathrm{y} \in \partial\mathscr{R}. \end{cases} \tag{4}$$

Although the integral equation (3) involves only the solution on the outer boundary, the effect of the holes is represented without approximation in the kernel function

$$g'(\mathrm{x}, \mathrm{y}, \xi) = -\frac{1}{2\pi} \left\{ \log|\mathrm{x}-\mathrm{y}| + \log|\mathrm{x}-\xi| \log|\mathrm{y}-\xi| \log a + \sum_{m=1}^{\infty} \frac{a^{2m}/m}{|\mathrm{x}-\xi|^m |\mathrm{y}-\xi|^m} \cos m[\theta(\mathrm{y}) - \theta(\mathrm{x})] \right\}. \tag{5}$$

The solution of equation (3) can be found by any one of the standard numerical methods for boundary integral equations [5]. This will determine the temperature and flux everywhere on $\partial\mathscr{R}$. The individual coefficients in the expansion (2) for the unknown boundary flux on the hole can then be recovered by quadrature from

$$q_0 = \frac{1}{a \log a} \left\{ \phi_0 + \int_{\partial\mathscr{R}} \left(\phi \frac{\partial g_0}{\partial n} - g_0 \frac{\partial \phi}{\partial n} \right) \mathrm{d}s \right\}, \tag{6}$$

$$q_{\lambda m} = \frac{1}{a} \left\{ m\phi_{\lambda m} + 2 \int_{\partial\mathscr{R}} \left(\phi \frac{\partial g_{\lambda m}}{\partial n} - g_{\lambda m} \frac{\partial \phi}{\partial n} \right) \mathrm{d}s \right\} \tag{7}$$

where

$$g_0 = -\frac{1}{2\pi} \log|\mathrm{x}-\xi|,$$

$$g_{1m} = \frac{a^m \sin m\theta}{2\pi|\mathrm{x}-\xi|^m}, \qquad g_{2m} = \frac{a^m \cos m\theta}{2\pi|\mathrm{x}-\xi|^m}. \tag{8}$$

Alternatively, the entire solution on the hole boundary can be recovered at once from

$$a\frac{\partial \phi}{\partial n}(\theta) = -\phi_0/\log a + \sum_{m=1}^{\infty} m(\phi_{1m} \sin m\theta + \phi_{2m} \cos m\theta) + \int_{\partial\mathscr{R}} \left(\phi \frac{\partial g_0'}{\partial n} - g_0' \frac{\partial \phi}{\partial n} \right) \mathrm{d}s \tag{9}$$

where

$$g_0'(\mathrm{x}, \xi) = \frac{1}{2\pi} \left\{ \log|\mathrm{x}-\xi|/\log a + \sum_{m=1}^{\infty} \frac{2a^m}{|\mathrm{x}-\xi|^m} \cos m[\theta(\mathrm{x}) - \theta] \right\}. \tag{10}$$

Perhaps equation (6) is the most practical result since the total flux from the hole is just $2\pi a q_0$.

Since the above procedure reduces the problem to a solution of an integral equation on the outer boundary alone, there is no need to discretize the solution on the boundary of the hole. Besides being more convenient, this also avoids certain conditioning problems, which are discussed in refs. [1, 2]. A corresponding set of integral equations can also be determined when flux is specified on the hole. These are recorded in ref. [2] but are not required for the examples considered in this paper.

3. AN INFINITE ROW OF HOLES IN A HALF-SPACE

As a first example, consider an infinite row of identical holes in a half-space (Fig. 2). Let a be the radius

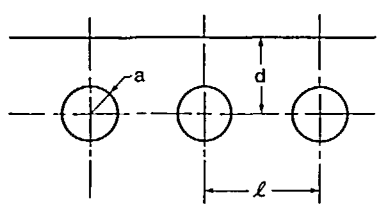

FIG. 2. A row of holes in a half-space.

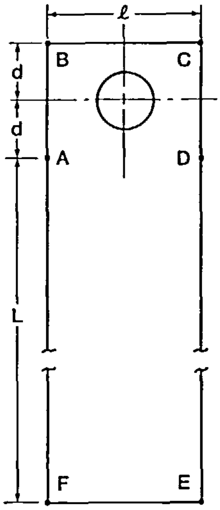

FIG. 3. Region for solving the single-hole equation (3) in the half-space problem.

of the holes, l the distance between their centers, and d their depth below the surface. The boundary of each hole has the same constant temperature ϕ_0, and the temperature on the surface of the half-space is taken to be zero. The standard handbook approximation for the total flux from any one hole in this problem is [4]

$$\phi_0/(aq_0) = \log\left[\frac{l}{\pi a}\sinh(2\pi d/l)\right]. \qquad (11)$$

The same result was also obtained from the solution of the zeroth-order equations in ref. [1] which assume that the heat flux is constant on the hole. The corresponding expression derived from the solution of the first-order equations, which retain up to the first harmonic in the boundary flux on the hole, is [1]

$$\phi_0/(aq_0) = \log\left[\frac{l}{\pi a}\sinh(2\pi d/l)\right] + \frac{\cosh^2(2\pi d/l)}{1 - \left(\frac{1}{3} + \frac{l^2}{\pi^2 a^2}\right)\sinh^2(2\pi d/l)}. \qquad (12)$$

In this section, we evaluate the accuracy of both approximations by comparing them to numerical solutions of the exact integral equation (3). Based on previous experience with this equation [3], numerical errors should be considerably less than 1%.

Since the solution is the same on every hole in the half-space, it suffices to solve equation (3) on the vertical strip shown in Fig. 3. The heat flux is zero on the sides of the strip and the semi-infinite region was closed at a large distance from the hole by another surface of zero flux. A distance of $L = 10 \times \max(2d, l)$ was found to be sufficient for practical convergence of the numerical solution. The boundary of the region was discretized as follows: segments AB, BC, and CD were each divided into 25 equal intervals, segments DE and FA into 50 equal intervals each, and segment EF into 5 equal intervals. The solution was assumed to be constant on each of these intervals, and the integral equation (3) was solved using a standard quadrature method [5]. The value of q_0 was recovered subsequently from equation (6). Results of this calculation are given in Table 1 for a range of d/a and l/a. The relative error of both equation (11) and equation (12) was computed with respect to these values and plotted in Figs. 4 and 5, respectively.

The handbook approximation (11) produces a sizable error unless the holes are widely spaced and/or moderately far from the surface (Fig. 4). On the other

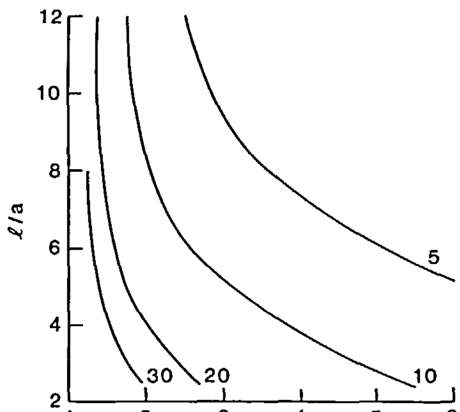

FIG. 4. Relative error (in percent) associated with the handbook approximation (11) for a row of holes in a half-space.

Table 1. Values of aq_0/ϕ_0 computed from equation (6) and numerical solutions of equation (3) for an infinite row of holes in a half-space

	l/a							
d/a	2.5	3.0	4.0	5.0	6.0	8.0	10.0	12.0
1.25	0.9884	1.064	1.169	1.238	1.285	1.342	1.373	1.391
1.5	0.5969	0.6611	0.7565	0.8225	0.8693	0.9287	0.9626	0.9832
2.0	0.3401	0.3878	0.4657	0.5248	0.5698	0.6309	0.6681	0.6918
2.5	0.2385	0.2758	0.3402	0.3923	0.4341	0.4944	0.5334	0.5594
3.0	0.1838	0.2142	0.2685	0.3144	0.3527	0.4108	0.4505	0.4778
4.0	0.1262	0.1482	0.1890	0.2254	0.2574	0.3905	0.3482	0.3767
5.0	0.09625	0.1135	0.1460	0.1758	0.2029	0.2489	0.2852	0.3134
6.0	0.07800	0.09205	0.1190	0.1442	0.1675	0.2083	0.2418	0.2690

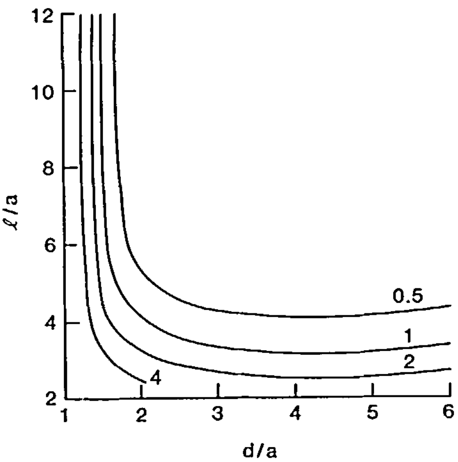

FIG. 5. Relative error (in percent) associated with the new result (12) for a row of holes in a half-space.

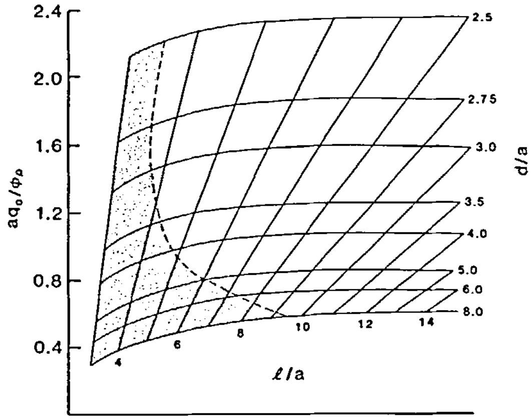

FIG. 7. Numerical results for a row of holes in a uniform slab with both surfaces at the same constant temperature.

hand, equation (12) gives an exceptionally accurate result even when the holes are closely spaced or very near the surface (Fig. 5).

Note in Fig. 5 that the error in equation (12) decreases with increasing depth until $d/a > 4$, and then begins to increase slightly. This is probably because the second harmonic in the representation for the heat flux on the hole boundary [which is neglected in equation (12)] becomes increasingly important as the depth increases relative to the hole spacing.

4. AN INFINITE ROW OF HOLES IN A SLAB

Next consider an infinite row of identical holes, equally spaced in the center of a uniform slab (Fig. 6). Let a be the radius of the holes, d the thickness of the slab, and l the distance between the centers of the holes. The boundary of each hole has the same constant temperature ϕ_0, and the temperature on at least one surface of the slab is taken to be zero. We consider two cases for the boundary condition on the other surface: (1) zero temperature and (2) zero heat flux.

In the first case, the standard handbook approximation for the total flux from any one hole is [4]

$$\phi_0/(aq_0) = \log\left[\frac{l}{\pi a}\sinh\,(\pi d/2l)\right]. \tag{13}$$

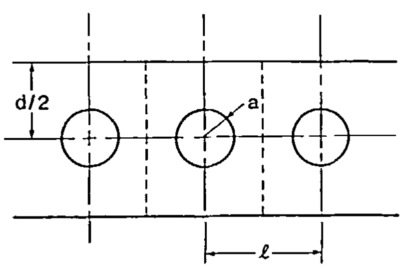

FIG. 6. A row of holes in a uniform slab. Dotted lines indicate the region for solving the single-hole equation (3).

A similar expression is not available for the second case. References to the origin of equation (13) in the literature are vague [6], and so it is difficult to determine its implicit assumptions. Unlike equation (11), equation (13) does not appear to be a solution of the zeroth-order equations in ref. [1].

Numerical solutions of the exact integral equation (3) were obtained on a single symmetric cell of the slab (showed dotted in Fig. 6). Each of the four sides was divided into 25 equal intervals and the solution obtained as before. The results for both cases are plotted in Figs. 7 and 8, and the error associated with equation (13) is shown in Fig. 9. In the shaded area of Fig. 7, the indicated numerical results differ by less than 5% from the approximation (13). Generally, greater errors are encountered with equation (13) as the slab gets thinner (Fig. 9). Rohsenow and Hartnett [4]

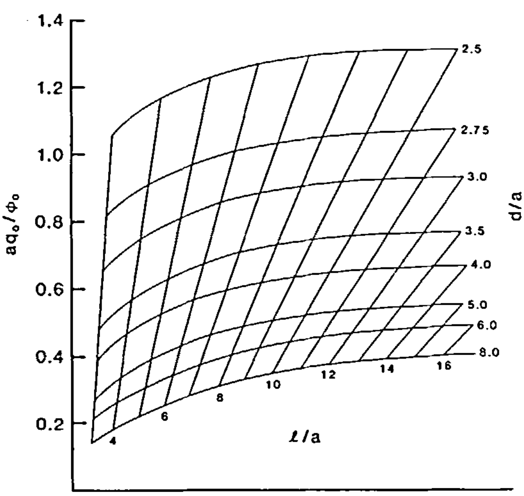

FIG. 8. Numerical results for a row of holes in a uniform slab with temperature constant on one surface and the other surface insulated.

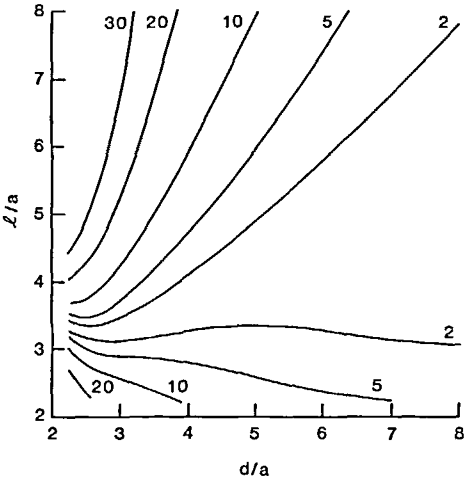

FIG. 9. Relative error (in percent) associated with the handbook expression (13) for a row of holes in a uniform slab.

qualify equation (13) with $d/a > 2$. If we impose $d/a > 5$, the error would be limited to 10%.

The effect of insulating one surface of the slab is not only to decrease the total flux from each hole but also to extend the range of influence of the spacing between the holes. In Fig. 7, the curves of constant d/a reach an effective asymptote when $l/a > 2d/a$, and so the hole spacing becomes unimportant when $l/d > 2$. In Fig. 8, this does not occur until $l/d > 3$.

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CONDUCTION STATIONNAIRE A PARTIR D'UNE RANGEE DE TROUS DANS UN DEMI-ESPACE OU UNE PLAQUE UNIFORME

Résumé—Une équation intégrale spéciale est utilisée pour analyser la conduction permanente à partir d'une rangée infinie de trous circulaires située sous la surface d'un demi-espace ou le long du plan médian d'une plaque uniforme. Des résultats numériques montrent que les approximations usuelles des manuels pour le flux thermique total à partir d'un trou quelconque peut donner des erreurs importantes dans les deux problèmes, même pour des valeurs modérées des paramètres du problème. Une approximation proposée récemment pour le problème du demi-espace offre une amélioration sensible de la précision.

STATIONÄRE WÄRMELEITUNG AN EINER UNENDLICH AUSGEDEHNTEN LOCHREIHE IN EINEM HALBRAUM ODER EINER EBENEN PLATTE

Zusammenfassung—Eine spezielle Randwert-Integralgleichung wird verwendet, um die stationäre Wärmeleitung an einer unendlich ausgedehnten Reihe von kreisförmigen Löchern, die unter der Oberfläche eines Halbraums oder in der Mittelebene einer ebenen Platte liegen, zu untersuchen. Die zahlenmäßigen Ergebnisse zeigen, daß die üblichen in Handbüchern empfohlenen Approximationen für den Gesamtwärmestrom, der von einem Loch ausgeht, in beiden Fällen zu erheblichen Fehlern führen können, selbst für kleine Werte der Einflußgrößen. Es wird gezeigt, daß eine kürzlich für den Halbraum vorgeschlagene Approximations-Alternative eine wesentliche Verbesserung der Genauigkeit bietet.

СТАЦИОНАРНАЯ ПЕРЕДАЧА ТЕПЛА ТЕПЛОПРОВОДНОСТЬЮ ОТ БЕСКОНЕЧНОГО РЯДА ОТВЕРСТИЙ В ПОЛУПРОСТРАНСТВЕ ОДНОРОДНОЙ ПЛИТЫ

Аннотация—Для анализа стационарной передачи тепла теплопроводностью от бесконечного ряда круглых отверстий, расположенных под поверхностью полупространства или вдоль средней плоскости однородной пластины, использовано решение специального граничного интегрального уравнения. Численные результаты показывают, что расчет суммарного потока тепла от любого из отверстий по представленным в справочниках общепринятым аппроксимациям может привести к значительным погрешностям в обеих задачах, даже при небольших значениях параметров. Показано, что аппроксимирующее соотношение, недавно предложенное для задачи полупространства, дает значительно более точные результаты.